NAMUNAVIY MASALALAR

1-masala.

$f : N \to N$ funksiyasi uchun:

$$f(1) = 1$$
$$f(n + 1) = f(n) + 2n \quad (n \geq 1)$$

$f(n)$ formulani toping.

Yechim

Ketma-ketlik farqini koʻramiz:

$$f(n + 1) - f(n) = 2n$$

$f(n)$ ni kvadrat koʻrinishda boʻladi deb taxmin qilamiz:

$$f(n) = a\,n^2 + b\,n + c$$

Farq:

$$f(n + 1) - f(n)$$
$$= a((n + 1)^2 - n^2) + b((n + 1) - n)$$
$$= a(2n + 1) + b$$

Bu $2n$ ga teng boʻlishi kerak:

$$a(2n + 1) + b = 2n$$

Koʻeffitsientlarni taqqoslaymiz:

$$2a = 2 \to a = 1$$

$$a + b = 0 \to 1 + b = 0 \to b = -1$$

Demak:

$$f(n) = n^2 - n + c$$

Endi $f(1) = 1$:

$$f(1) = 1^2 - 1 + c = 1 \;\to\; 0 + c = 1 \;\to\; c = 1$$

Javob:

$$f(n) = n^2 - n + 1$$

2-masala.

$f : Z \to Z$ uchun:

$$f(0) = 0$$
$$f(x + 2) = f(x) + 4x + 2 \quad (x \in Z)$$

$f(x)$ ni toping.

Yechim

$f(x)$ polinom, deb olaylik:

$$f(x) = a\,x^2 + b\,x + c$$

Hisoblaymiz:

$$\begin{aligned} f(x + 2) &= a(x + 2)^2 + b(x + 2) + c \\ &= a(x^2 + 4x + 4) + bx + 2b + c \\ &= a\,x^2 + 4a\,x + 4a + b\,x + 2b + c \end{aligned}$$

Farq:

$$f(x + 2) - f(x)$$
$$= (a\,x^2 + 4a\,x + 4a + b\,x + 2b + c) - (a\,x^2 + b\,x + c)$$
$$= 4a\,x + 4a + 2b$$

Bu shartga ko‘ra $4x + 2$ ga teng:

$$4a\,x + (4a + 2b) = 4x + 2$$

Demak:

$$4a = 4 \to a = 1$$

$$4a + 2b = 2 \to 4 + 2b = 2 \to 2b = -2 \to b = -1$$

Shunday qilib:

$$f(x) = x^2 - x + c$$

$f(0) = 0$ dan:

$$c = 0$$

Javob:

$$f(x) = x^2 - x$$

3-masala. $x + y$ va $x - y$ bilan (Yensen)

Masala.
$f : R \to R$ funksiya polinom bo‘lib, quyini qanoatlantirsin:

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \quad (x, y \in R)$$

$f(x)$ ni toping.

Yechim

Polinom sifatida:

$$f(x) = a\,x^2 + b\,x + c$$

Hisoblaymiz:

1. $f(x + y)$:

$$a(x + y)^2 + b(x + y) + c$$

2. $f(x - y)$:

$$a(x - y)^2 + b(x - y) + c$$

Yig‘indisi:

$$\begin{gathered} f(x + y) + f(x - y) \\ = a(+ (x - y)^2) + b((x + y) + (x - y)) + 2c \\ = a(2x^2 + 2y^2) + b(2x) + 2c \\ = 2a\,x^2 + 2a\,y^2 + 2b\,x + 2c \end{gathered}$$

O‘ng tomon:

$$\begin{gathered} 2f(x) + 2f(y) \\ = 2(ax^2 + bx + c) + 2(ay^2 + by + c) \\ = 2a\,x^2 + 2a\,y^2 + 2b\,x + 2b\,y + 4c \end{gathered}$$

Taqqoslaymiz:

y uchun chiziqli had: chapda 0, o‘ngda $2b\,y \to b = 0$

c uchun: $2c = 4c \rightarrow 2c = 0 \rightarrow c = 0$

Demak:

$$f(x) = a\,x^2$$

istalgan $a \in R$ bo‘lishi mumkin.

Javob:

$f(x) = a\,x^2$ (a — ixtiyoriy haqiqiy son)

4-masala. Involyutiv funksiya (kompozitsiya)

Masala.
$f : R \rightarrow R$ funksiya **qat’iy o‘suvchi** va

$$f(f(x)) = x \quad (x \in R)$$

bo‘lsin. $f(x)$ ni toping.

Yechim

$f(f(x)) = x$ bo‘lsa, f o‘zining **teskari funksiyasi**:

$$f^{-1}(x) = f(x)$$

Faraz qilaylik, $f(x_0) \neq x_0$.

1. Agar $f(x_0) > x_0$ bo‘lsa, qat’iy o‘suvchi bo‘lganligi uchun:

$$x_0 < f(x_0) \rightarrow f(x_0) < f(f(x_0)) = x_0$$

Bu esa $x_0 < x_0$ bo‘lib qoladi — **zidlik**.

2. Agar $f(x_0) < x_0$ bo‘lsa, yana o‘suvchi bo‘lgani uchun:

$$f(x_0) < x_0 \rightarrow f(f(x_0)) < f(x_0) \rightarrow x_0 < f(x_0)$$

Bu ham zidlik.

Demak, ikkala holat ham mumkin emas, yagona imkoniyat:

$f(x) = x$ barcha x uchun

Javob:

$$f(x) = x$$

5-masala.

$f : Z \to Z$ funksiyasi uchun:

$$f(0) = 0$$
$$f(n + 1) = 2f(n) + 1 \quad (n \in Z)$$

$f(n)$ ni toping (butun n uchun).

Yechim

Ketma-ket qiymatlarni ko‘ramiz:

$$f(0) = 0$$
$$f(1) = 2 \cdot 0 + 1 = 1$$
$$f(2) = 2 \cdot 1 + 1 = 3$$
$$f(3) = 2 \cdot 3 + 1 = 7$$
$$f(4) = 2 \cdot 7 + 1 = 15$$

Ko‘rinib turibdi:

$$f(n) = 2^n - 1$$

deb taxmin qilamiz ($n \geq 0$ uchun).

Induksiya:

Bazaviy: $n = 0$

$f(0) = 0,\ 2^0 - 1 = 0 \ \to$ rost

Faraz qilaylik n uchun rost:

$$f(n) = 2^n - 1$$

Unda:

$$\begin{aligned} f(n + 1) &= 2f(n) + 1 \\ &= 2(2^n - 1) + 1 \\ &= 2^{n+1} - 2 + 1 \\ &= 2^{n+1} - 1 \end{aligned}$$

demak $n + 1$ uchun ham rost.

✅ **Javob (n ≥ 0):**

$$f(n) = 2^n - 1$$

(agar manfiy n lar kiritilsa, alohida ko‘rib chiqish kerak; o‘rtacha daraja uchun $n \geq 0$ yetarli).

6-masala.

Masala.
$f : R \to R$ uchun:

$f(x) - f(y) = x^2 - y^2$ (barcha $x, y \in R$)

$f(x)$ ni toping.

Yechim

$y = 0$ qo‘yamiz:

$$f(x) - f(0) = x^2 - 0$$

demak:

$$f(x) = x^2 + C$$

bu yerda $C = f(0)$.

Tekshiramiz:

$$f(x) - f(y) = (x^2 + C) - (y^2 + C) = x^2 - y^2$$

tenglama qanoatlanadi.

Javob:

$f(x) = x^2 + C$ (C — ixtiyoriy haqiqiy son)

7-masala.

$f : R \to R$ funksiya uchun:

$f(2x) = 4 f(x)$ (barcha $x \in R$)

$$f(1) = 3$$

$f(x)$ ni toping. Polinom yechimni toping.

Yechim

Polinom yechimni $a\,x^2$ ko‘rinishda izlaymiz:

$$f(x) = a\,x^2$$

Tekshiramiz:

$$f(2x) = a\,(2x)^2 = 4\,a\,x^2$$
$$4\,f(x) = 4\,(a\,x^2) = 4\,a\,x^2$$

Teng.

Endi $f(1) = 3$:

$$a \cdot 1^2 = 3 \rightarrow a = 3$$

Javob:

$$f(x) = 3\,x^2$$

8-masala.

$f : R \rightarrow R$ polinom bo‘lib:

$$f(x) + f(1 - x) = x^2 + (1 - x)^2$$

tenglikni qanoatlantirsin. $f(x)$ ni toping.

Yechim

$f(x)$ ni umumiy kvadrat polinom deb yozamiz:

$$f(x) = a\,x^2 + b\,x + c$$

$$f(1 - x):$$

$$\begin{aligned} f(1 - x) &= a(1 - x)^2 + b(1 - x) + c \\ &= a(1 - 2x + x^2) + b - b\,x + c \end{aligned}$$

Yig‘indini hisoblaymiz:

$$f(x) + f(1 - x)$$
$$= (a\,x^2 + b\,x + c) + [a(1 - 2x + x^2) + b - b\,x + c]$$

$$= a\,x^2 + a\,x^2 + a(1 - 2x) + b\,x - b\,x + c + b + c$$
$$= 2a\,x^2 - 2a\,x + a + (2c + b)$$

O‘ng tomoni:

$$x^2 + (1 - x)^2 = x^2 + (1 - 2x + x^2) = 2x^2 - 2x + 1$$

Taqqoslaymiz:

- x^2 koeffitsienti: $2a = 2 \rightarrow a = 1$
- x koeffitsienti: $-2a = -2$ → mos, chunki $a = 1$
- erkin had: $a + 2c + b = 1 \rightarrow 1 + 2c + b = 1 \rightarrow 2c + b = 0$

Demak:

$$f(x) = x^2 + b\,x + c$$

va $2c + b = 0$. Masalan $c = t$ deb olsak, $b = -2t$.

Javob :

$f(x) = x^2 - 2t\,x + t$ (t — ixtiyoriy haqiqiy son)

9-masala.

Masala.
$f : Z \rightarrow Z$ funksiyasi:

$$f(0) = 0$$
$$f(n + 1) = f(n) + 2n + 1 \quad (n \in Z)$$

ni qanoatlantiradi. $f(n)$ ni toping ($n \geq 0$ uchun).

Yechim

E’tibor bering:

$$2n + 1 = (n + 1)^2 - n^2$$

Shuning uchun:

$$f(n + 1) - f(n) = (n + 1)^2 - n^2$$

Bu n^2 bilan uyg‘un, shuning uchun $f(n) = n^2$ bo‘lishi mumkin deb taxmin qilamiz.

Tekshiramiz:

Faraz:

$$f(n) = n^2$$

Unda:

$$f(n+1) - f(n) = (n+1)^2 - n^2 = 2n + 1$$

berilgan tenglikka mos.

Bazaviy:

$$f(0) = 0^2 = 0$$

Javob (n ≥ 0):

$$f(n) = n^2$$

10-masala.

$f : R \to R$ polinom bo‘lib:

$f(x+1) + f(x-1) = 2f(x) + 4$ (barcha $x \in R$)

tenglikni qanoatlantiradi. $f(x)$ ni toping.

Yechim

Polinom shakl:

$$f(x) = a\,x^2 + b\,x + c$$

Hisoblaymiz:

1. $f(x+1)$:

$$\begin{gathered} a(x+1)^2 + b(x+1) + c \\ = a(x^2 + 2x + 1) + b\,x + b + c \\ = a\,x^2 + 2a\,x + a + b\,x + b + c \end{gathered}$$

2. $f(x-1)$:

$a(x-1)^2 + b(x-1) + c = a(x^2 - 2x + 1) + b\,x - b + c$
$= a\,x^2 - 2a\,x + a + b\,x - b + c$

Yig‘indisi:

$$\begin{aligned} f(x+1)+f(x-1) &= (a\,x^2+2a\,x+a+b\,x+b+c) \\ &\quad + (a\,x^2-2a\,x+a+b\,x-b+c) \\ &= 2a\,x^2+2b\,x+(2a+2c) \end{aligned}$$

O‘ng tomon:

$$2\,f(x)+4 = 2(a\,x^2+b\,x+c)+4 = 2a\,x^2+2b\,x+2c+4$$

Taqqoslaymiz:

$$2a\,x^2+2b\,x+2a+2c = 2a\,x^2+2b\,x+2c+4$$

Demak:

$$2a+2c = 2c+4 \ \rightarrow\ 2a = 4 \ \rightarrow\ a = 2$$

b va c erkin, chunki ular bekor bo‘lib ketdi.

Javob:

$f(x) = 2\,x\text{^}2 + b\,x + c$ (b, c — ixtiyoriy haqiqiy sonlar)